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On The Positive Pell Equation

$$y^2 = 120x^2 + 4$$

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Abstract – The binary quadratic equation represented by the positive pellian $y^2 = 120x^2 + 4$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Index Terms – Binary quadratic, hyperbola, parabola, integral solutions, pell equation.
2010 mathematics subject classification : 11D09.

1. INTRODUCTION

The binary quadratic equation of the form $y^2 = Dx^2 + 1$, where D is non- square positive integer, has been studied by various mathematicians for its non- trivial integral solutions when D takes different integral values [1-4]. In [5] infinitely many Pythagorean triangle in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation of $y^2 = 3x^2 + 1$. In [6], a special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$. In [7], different patterns of infinitely many Pythagorean triangles are obtained by employing the non-trivial solutions of $y^2 = 12x^2 + 1$. In this context one may also refer [8-16]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 120x^2 + 4$ representing a hyperbola. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under the consideration, a special Pythagorean triangle is obtained.

2. METHOD OF ANALYSIS

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 120x^2 + 4 \qquad (1)$$

whose smallest positive integer solution is

$$x_0 = 2, y_0 = 22$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 120x^2 + 1$$

whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{120}} g_n, \tilde{y}_n = \frac{1}{2} f_n$$

Where

$$f_n = (11+\sqrt{120})^{n+1} + (11-\sqrt{120})^{n+1}$$

$$g_n = (11+\sqrt{120})^{n+1} - (11-\sqrt{120})^{n+1}$$

Applying Brahmagupta lemma between (x_0,y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = f_n + \frac{11}{\sqrt{120}} g_n$$

$$y_{n+1} = 11 f_n + \sqrt{120} g_n$$

The recurrence relations satisfied by x and y are given by

$$2x_{n+1} - 44x_{n+2} + 2x_{n+3} = 0$$

$$y_{n+1} - 22y_{n+2} + y_{n+3} = 0$$

Some numerical examples of x and y satisfying (1) are given in the following table below:

n	x_{n+1}	y_{n+1}
-1	2	22
0	44	482
1	966	10582
2	21208	232322
3	465610	5100502

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_n and y_n are always even.

2. Each of the following expressions is a nasty number

- $6(11y_{2n+2} - 120x_{2n+2} + 2)$
- $6(22y_{2n+2} - y_{2n+3} + 2)$
- $6(11y_{2n+2} - 120x_{2n+2} + 2)$

3. $(11y_{3n+3} - 120x_{3n+3}) - 3\left[(11+\sqrt{120})^{n+1} + (11-\sqrt{120})^{n+1}\right]$ is a cubical integer.

4. $x_{n+3} = 22x_{n+2} - x_{n+1}$

5. $y_{n+1} = x_{n+2} - 11x_{n+1}$

6. $y_{n+2} = 11x_{n+2} - x_{n+1}$

7. $y_{n+3} = 241x_{n+2} - 11x_{n+1}$

8. $241x_{n+2} = 11x_{n+3} - y_{n+1}$

9. $11x_{n+2} = x_{n+3} - y_{n+2}$

10. $x_{n+2} = 11x_{n+3} - y_{n+3}$

11. $240x_{n+2} = y_{n+3} - y_{n+1}$

12. $120x_{n+2} = y_{n+3} - 11y_{n+2}$

13. $241x_{n+1}^{2} = x_{n+3}x_{n+1} - 22y_{n+1}x_{n+1}$

14. $241x_{n+1}y_{n+1} = x_{n+3}y_{n+1} - 22y_{n+1}^{2}$

15. $120x_{n+2}x_{n+1} = y_{n+3}x_{n+1} - 11y_{n+2}x_{n+1}$

16. $y_{n+3} = 22y_{n+2} - y_{n+1}$

17. $120x_{n+1} = y_{n+2} - 11y_{n+1}$

18. $120x_{n+2} = 11y_{n+2} - y_{n+1}$

19. $120x_{n+3} = 241y_{n+2} - 11y_{n+1}$

20. $241y_{n+2} = 11y_{n+3} - 120x_{n+1}$

21. $11y_{n+2} = y_{n+3} - 120x_{n+2}$

22. $y_{n+2} = 11y_{n+3} - 120x_{n+3}$

23. $120y_{n+2} = 1320x_{n+2} - 120x_{n+1}$

24. $120y_{n+1}y_{n+2} = 1320y_{n+1}x_{n+2} - 120x_{n+1}y_{n+1}$

25. $240y_{n+2}^{2} = 120x_{n+3}y_{n+2} - 120x_{n+1}y_{n+2}$

3. REMARKABLE OBSERVATIONS

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table below:

S.No	Hyperbola	x, y
1.	$x^2 - 120y^2 = 4$	$11y_{n+1} - 120x_{n+1}$, $\sqrt{120}(11x_{n+1} - y_{n+1})$
2.	$120x^2 - y^2 = 480$	$22y_{n+1} - y_{n+2}$, $\frac{1}{\sqrt{120}}(11y_{n+2} - 241y_{n+1})$
3.	$120x^2 - y^2 = 480$	$483y_{n+2} - 22y_{n+3}$, $\frac{1}{\sqrt{120}}(241y_{n+3} - 5291y_{n+2})$
4.	$120x^2 - y^2 = 480$	$11y_{n+1} - 120x_{n+1})$, $\frac{1}{\sqrt{120}}(1320x_{n+1} - 120y_{n+1})$
5.	$120x^2 - y^2 = 232320$	$\frac{1}{22}(483y_{n+1} - y_{n+3})$, $\frac{1}{22\sqrt{120}}(11y_{n+3} - 5291y_{n+1})$
6.	$120x^2 - y^2 = 480$	$11x_{n+2} - 241x_{n+1}$, $\frac{1}{\sqrt{120}}(2640x_{n+1} - 120x_{n+2})$
7.	$120x^2 - y^2 = 480$	$241x_{n+3} - 5291x_{n+2}$, $\frac{1}{\sqrt{120}}(57960x_{n+2} - 2640x_{n+3})$
8.	$120x^2 - y^2 = 3345408000$	$\frac{1}{2640}(1320x_{n+3} - 634920x_{n+1})$, $\frac{1}{2640\sqrt{120}}(6955200x_{n+1} - 14400x_{n+3})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the table below:

S.No	Parabola	x, y
1.	$x \quad - 120y^2 = 4$	$(11y_{n+1} - 120x_{n+1}) + 2$, $\sqrt{120}(11x_{n+1} - y_{n+1})$

2.	$120x - y^2 = 480$	$(22y_{n+1} - y_{n+2}) + 2,$ $\frac{1}{\sqrt{120}}(11y_{n+2} - 241y_{n+1})$
3.	$120x - y^2 = 480$	$(483y_{n+2} - 22y_{n+3}) + 2,$ $\frac{1}{\sqrt{120}}(241y_{n+3} - 5291y_{n+2})$
4.	$120x - y^2 = 480$	$(11y_{n+1} - 120x_{n+1}) + 2,$ $\frac{1}{\sqrt{120}}(1320x_{n+1} - 120y_{n+1})$
5.	$2640x - y^2 = 232320$	$\frac{1}{22}(483y_{n+1} - y_{n+3}) + 2,$ $\frac{1}{22\sqrt{120}}(11y_{n+3} - 5291y_{n+1})$
6.	$120x - y^2 = 480$	$(11x_{n+2} - 241x_{n+1}) + 2,$ $\frac{1}{\sqrt{120}}(2640x_{n+1} - 120x_{n+2})$
7.	$120x - y^2 = 480$	$(241x_{n+3} - 5291x_{n+2}) + 2,$ $\frac{1}{\sqrt{120}}(57960x_{n+2} - 2640x_{n+3})$
8.	$316800x - y^2 = 3345408000$	$\frac{1}{2640}(1320x_{n+3} - 634920x_{n+1}) + 2,$ $\frac{1}{2640\sqrt{120}}(6955200x_{n+1} - 14400x_{n+3})$

3. Consider $p = x_{n+1} + y_{n+1}$, $q = x_{n+1}$. Observe that p>q>0.Treat p, q as the generators of the Pythagorean triangle T(α, β, γ),where $\alpha = 2pq, \beta = p^2 - q^2$, $\gamma = p^2 + q^2$. Let A, P represent the area and perimeter of T(α, β, γ).Then the following interesting relations are observed.

a) $\alpha + 59\gamma - 60\beta = -4$

b) $6(\beta - \frac{4A}{P})$ is a nasty number.

c) $6(2\alpha - \frac{4A}{P} + \beta)$ is a nasty number.

4. CONCLUSION

In this paper , we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2 = 120x^2 + 4$. As the binary quadratic Diophantine equations are rich in variety , one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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